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On the symbolic dynamics of the Henon map

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Abstract. We study a recently proposed method for finding all periodic orbits of the dissipative Henon map. For the 'canonical' values $a = 1.4$, $b = 0.3$ this method is found to give indeed the good symbolic dynamics proposed earlier by two of us, but it does not work as proposed for larger values of b. This is seen in two ways. Although the method is thus not universally valid, it can be used to study problems related to periodic orbits. We find that (for the usual values $a = 1.4$, $b = 0.3$) expressions for average quantities are not easily obtained precisely from periodic orbits, in agreement with recent claims.

1. Introduction

Being one of the simplest non-trivial dissipative systems showing chaotic behaviour, the Henon map $(x, y) \to (a + by - x^2, x)$ is often used as a testing ground in dynamical systems theory like the Ising model in statistical mechanics. It is now proven [1] that there is a set of finite measure in the control parameters a and b for which the map has a strange attractor. But the proof covers only very small values of the constant b, and does not include the classical values $a = 1.4, b = 0.3$ studied in the original paper by Henon [2] and in most subsequent papers.

For $b = 0$, the Henon map reduces to the logistic map $x \to a - x^2$. Superficially, these two systems are rather similar. But while the logistic map has at most one chaotic attractor for any given value of a, the Henon map can have arbitrarily many [3]. Thus, studying its symbolic dynamics might seem rather hopeless. Technically, the main problem is that the Henon map is not uniformly hyperbolic, thus creating both mathematical [3] and numerical [4] problems.

Nevertheless, there are a number of results which—though mostly not proven—are believed to be true.

The first result is that there exists a binary symbolic dynamics in cases where there is no chaotic attractor, and where the unstable orbits form a horseshoe. This was proven in [5], but it is not very surprising as the dynamics on horseshoes is always comparatively simple.

A less trivial (but only conjecured) result was obtained by two of us [6], when we found binary partitions which seemed to be generating partitions also when there is a strange attractor. The guiding principle therein was that a good partition should pass through all 'primary' tangencies between stable and unstable manifolds. This is the most straightforward generalisation of the fact that generating partitions for the logistic map pass through the critical point. The resulting partitions for the cases $(a, b) = (1.4, 0.3)$ and $(a, b) = (1.0, 0.54)$ are reproduced in figures 1 and 2. Their correctness was verified

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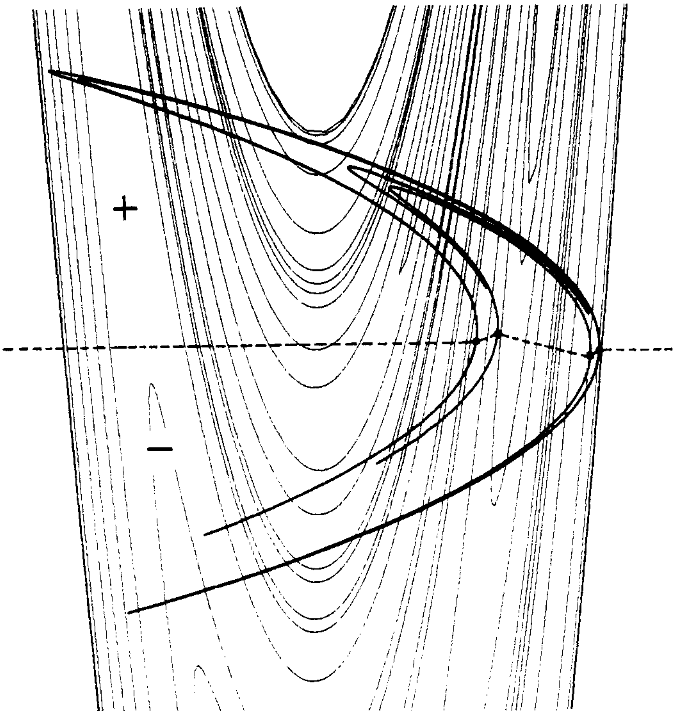

Figure 1. The Henon attractor for $(a, b) = (1.4, 0.3)$ together with part of the stable manifold of the fixed point at $x = y = 0.883\ldots$. Also shown is the generating partition proposed in [6]. It passes through the 'principal' tangency points between stable and unstable manifolds. A tangency point is called principal if the sum of the curvatures of both manifolds at this point is smaller than at any of its images and pre-images.

by comparing the metric entropy h with the Lyapunov exponent λ. Let us denote by $p(s_1, \ldots, s_N)$ the weight of the cylinder [7] $S = (s_1 \ldots s_N)$, and by

$$H_N = -\sum_S p(S) \ln p(S) \tag{1.1}$$

the block entropies representing the average information needed to specify a symbol sequence of length N. The differences

$$h_N = H_{N+1} - H_N \tag{1.2}$$

converge from above to the metric entropy,

$$h_N \geq h_{N+1} \geq 0 \tag{1.3}$$

$$h = \lim_{N \to \infty} h_N. \tag{1.4}$$

Since by Pesin's identity $h = \lambda$, we must have

$$h_N \geq \lambda \tag{1.5}$$

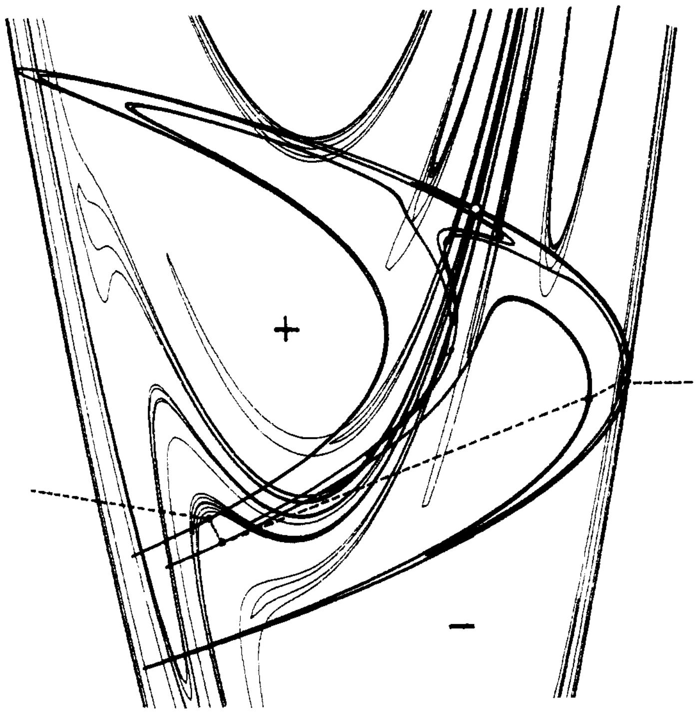

Figure 2. As figure 1, but for $(a, b) = (1.0, 0.54)$.

for all N. For a wrong symbolic representation, (1.5) would be violated for sufficiently large N.

The next important step was taken in [8], where the partition of [6] was shown to lead to a rather simple grammar. Since the topological entropy of the Henon map is less than $\ln 2$, some sequences must be 'pruned'. In one-humped maps of the interval, the rule for this pruning is simple [9]: first, an ordering in the set of symbol sequences is defined, and then all sequences are forbidden for which any shift is larger than the 'kneading sequence', which is the sequence corresponding to the orbit starting at the critical point. According to [8], the pruning in the case of the Henon map is completely analogous. We have just to replace the single kneading sequence by a set of sequences corresponding to orbits starting at primary tangencies. The ordering of the primary tangencies on the broken curves in figures 1 and 2 corresponds then also to an ordering of kneading sequences, called the pruning front. While the correctness of this was verified in [8] for periodic orbits up to length 18, one of us [10] verified it also with much higher precision for chaotic orbits.

For $(a, b) = (1.4, 0.3)$, the monotonicity of the pruning front was first checked by computing 120 primary tangency points to very high precision ($\geq$ 14 decimal digits). When ordered by their x coordinate, their kneading sequences were indeed monotonically increasing in the symbol sequence ordering of [8], and they indeed form a monotonic pruning front. Next, we used the partition thus defined to verify that the cylinders of length 22 visited by a very long chaotic trajectory are exactly those allowed by the pruning. Finally, we obtained the grammar for all allowed cylinders of length 18 in terms of a list of forbidden words (see table 1).

In neither [8] nor [10] was it verified for the less trivial case of large values of b,

Table 1. List of forbidden words in the symbolic dynamics of the Henon map with $a = 1.4$ and $b = 0.3$.

Length	Forbidden word
4	0000
4	0010
4	0110
7	0101000
7	0111000
8	01111000
8	10011100
8	10111100
9	000111100
9	100111011
9	101111011
11	00111001111
11	01111001111
12	100111010100
12	101111010100
12	010100111100
12	011100111100
12	000111101111
13	0001111011101
13	0101001111011
13	0111001111011
14	01010011110101
14	01110011110101
15	010100111101001
15	011100111101001
16	0111010001111011
16	0111110001111011
17	00011110111001110
17	11001110101011110
17	11011110101011110
17	01111010001111011
17	01111110001111011
18	011101000111101010
18	011111000111101010
18	101110100011110101
18	101111100011110101

such as in figure 2.

A final result on topological dynamics of the Henon map was given in a recent paper by Biham and Wenzel [11]. They claimed to be able to obtain in a systematic way all periodic orbits of the Henon map, in such a way that together with the orbits they also get the correct symbolic dynamics. More precisely, they proceed as follows. In order to obtain an orbit with period p, they consider a set of p variables $\boldsymbol{x}(t) = x_1(t), \ldots, x_p(t)$ with periodic boundary conditions, $x_{p+1}(t) = x_1(t)$. For these, they assume a set of coupled differential equations

$$\mathrm{d}x_i/\mathrm{d}t = s_i f_i(\boldsymbol{x}) \tag{1.6}$$

with s_i constants and with

$$f_i(\boldsymbol{x}) = x_{i+1} - a - bx_{i-1} + x_i^2. \tag{1.7}$$

It is obvious that $f_i = 0$ if the x_i form an orbit of the Henon map. Thus, periodic orbits of the Henon map correspond to stationary solutions of (1.6) and vice versa. By searching in the space of couplings s_i for stable fixed points of (1.6), one can thus hope to find all periodic orbits (this method for solving nonlinear equations is indeed very old, see e.g. [12]).

The surprising claims of [11], based on purely numerical grounds without any theoretical justification, are the following.

(i) A complete search in the s_i is not needed. All periodic orbits are found with s_i either 1 or -1, i.e. one only has to search in the space of 2^N binary strings of length N.

(ii) For each symbol sequence $S = s_1, ..., s_p$ there exists at most one stable fixed point of (1.6), and all these fixed points have a common basin of attraction, characterised by $|x_i(0)| \approx \sqrt{a}$.

(iii) Inversely, no two different sequences S_1 and S_2 can give rise to the same fixed point. Since periodic orbits are dense on the Henon attractor, this means in particular that the sequences S define a generating partition and a good symbolic dynamics. For notational convenience, we shall in the following use the symbols 0 and 1 instead of -1 and 1.

(iv) System (1.6) has only fixed points as bounded solutions, i.e. it has no limit cycles and no strange attractors. Thus if a solution of (1.6) with $s_i = \pm 1$ does not escape to infinity, it goes to a periodic orbit of the Henon map; if it does escape, then there is no orbit with this symbol sequence.

If these claims were correct, they would be most useful. They would suggest that symbolic dynamics could be easily found along similar lines also for other systems like the conservative Henon map or the standard map [13]. In these cases no generating partitions are known as yet. Also, they would much ease the use of periodic orbits to estimate characteristic quantities of strange attractors like Lyapunov exponents, entropies, dimensions, and $f(\alpha)$ spectra [14]. In [8, 15], it was claimed that unstable periodic orbits are not only a useful theoretical tool, but that they can also be used very efficiently to obtain numerical estimates for these characteristics.

In the next section, we shall test the claims of [11]. In section 3, we shall use their results to test the numerical convergence of the Lyapunov exponent and of generalised dimensions.

2. Attractors of equation (1.6)

Let us first check the claims of [11] for the conventional parameters $a = 1.4$, $b = 0.3$ which represent essentially the small-b (and thus the large dissipation) case.

In [11], all fixed points of (1.6) with p up to 28 were obtained by Runge–Kutta integration with step size $dt = 0.1$. The numbers $N(p)$ of fixed points agreed up to $p = 12$ with the numbers of periodic orbits of the Henon map given in [8] (for $p > 12$, no values were quoted in [8]). The estimate for the topological entropy obtained from these numbers:

$$k = \lim_{p \to \infty} k_p \simeq 0.4650 \pm 0.0002 \tag{2.1}$$

$$k_p = (1/p) \ln(N(p) - 1) \tag{2.2}$$

agreed reasonably well with the most precise previous value $k = 0.4630$ [6, 19].

We checked that the same numbers $N(p)$ were indeed obtained with a simple Euler method with step sizes up to $dt = 0.47$, provided all $x_i(0)$ were taken to be ≈ 0.8. We also verified that—with these initial conditions—there were no limit cycles and no strange attractors.

Finally, we checked that the induced symbolic dynamics coincides with that of [6]. All sequences $S = s_1, \ldots, s_p$ agreed exactly with the symbol sequences obtained according to the partition given in [6], and no sequence forbidden according to table 1 did indeed occur.

These results suggest that for $a = 1.4$ and $b = 0.3$ the method of [11] gives indeed all periodic orbits with the correct symbolic dynamics. The absence of limit cycles and more complicated limit sets suggests furthermore that there should exist a Lyapunov function for (1.6). For $b = 0$ (i.e. for the logistic map), we have been able to prove that Lyapunov functions exist at least locally. Details are given in the appendix. For $b \neq 0$ we have no idea how a Lyapunov function should look. We might mention at this point that the Hamiltonian system mentioned in [11] is of no relevance to these results.

In order to see whether the above results also hold for larger values of b, let us now turn to the case $(a, b) = (1.0, 0.54)$.

While we found only fixed points as attractors of (1.6) for $p \leq 7$, we found a limit cycle for $p = 8$, for the symbol sequence $S = 11111010$. By using different integration methods with different step sizes, we verified that this was not a numerical artefact. Also, we verified that near the centre of the limit cycle there was an unstable fixed point. Additional limit cycles were found for $p = 16$, $p = 20$ and $p = 22$.

Our next finding contradicts point (iii) above. For period 22 we found that the same periodic orbit with $x_1 = -0.227\,626$ and $x_2 = 0.179\,386$ is a stationary solution of (1.6) for two different symbol sequences, namely $S_1 = 0011101011111010111010$ and $S_2 = 1111101011111010111010$. Thus, the method does not give a good symbolic dynamics, and simply counting the total number of fixed points of (1.6) does not necessarily give the correct topological entropy.

This already casts doubt on the generality of the method of [11]. But it might be that one should simply disregard the limit cycles, with the fixed points still giving all periodic orbits of the Henon map. Also, it might be that one gets still a good partition, if one takes into account only the symbol sequence S with the fastest convergence, if more than one sequence leads to the same fixed point.

Since we have no reliable alternative method of finding all periodic orbits of the Henon map, a direct check is not easy. But we can again compare the induced symbolic dynamics with that of [6]. This time we find that the two are not the same. In figure 3, we show all periodic points up to a given period, as obtained from (1.6). We represent (x_{i+1}, x_i) by a triangle if $s_i = -1$, and by a cross if $s_i = 1$. Comparing with figure 2, we see a definite disagreement.

To compare the two partitions indicated in figures 2 and 3 by broken curves, we calculated the metric entropies by estimating the weights of cylinder sets in a random aperiodic trajectory. From the plot in figure 4 of the entropy estimates h_n (see equation (1.2)), we cannot prefer either partition. Both seem to converge to a value in agreement with the Lyapunov exponent $\lambda = 0.263\,15 \pm 0.000\,02$.

Indeed, the difference between figures 2 and 3 does not mean that the two symbolic dynamics are incompatible with each other. Instead, we found that one can be translated into the other (within the limits of our accuracy) by a somewhat tricky recoding.

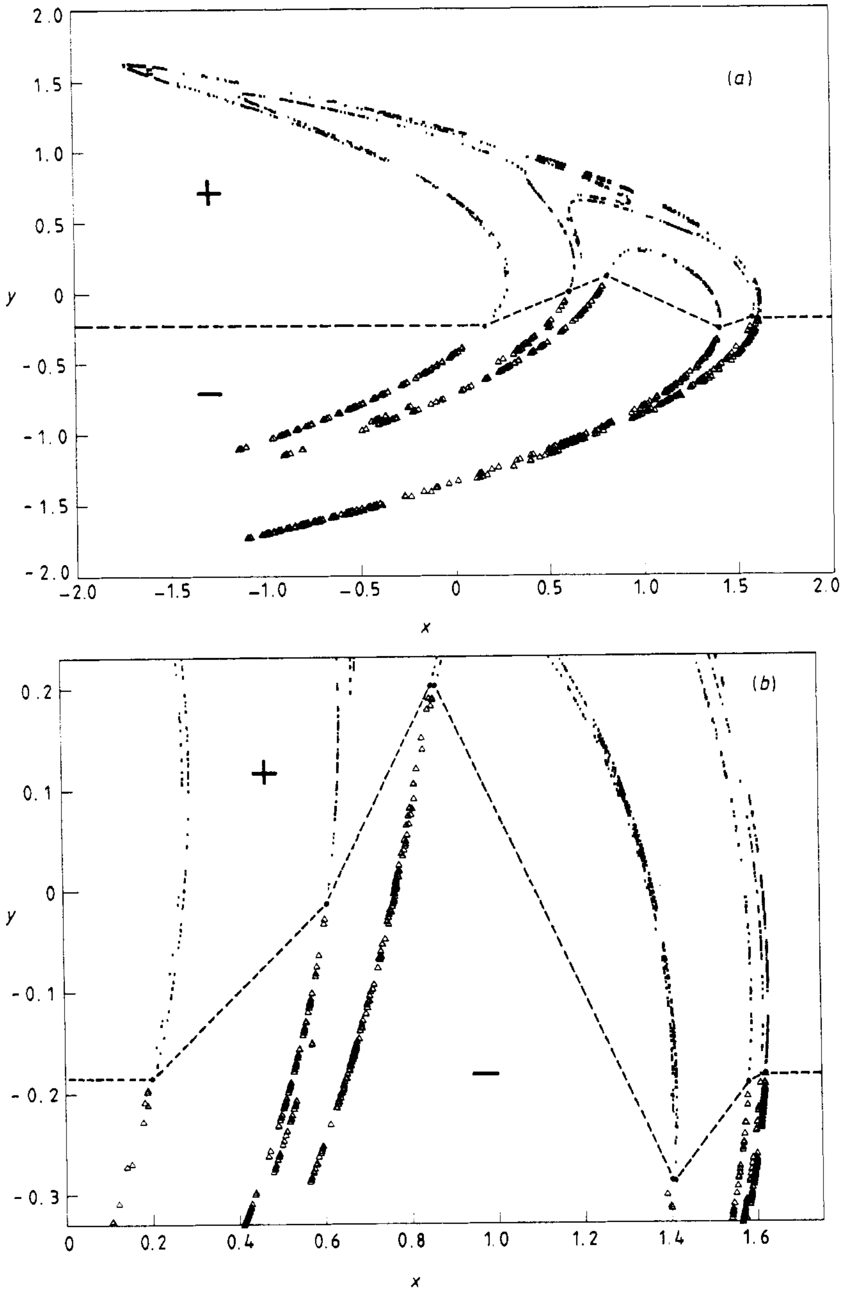

Figure 3. (*a*) Periodic points of the Henon map with $a = 1.0, b = 0.54$, obtained from (1.6) with $p = 21$. The broken curve indicates the partition induced by the symbol sequences $s_1 \ldots s_p$. Plots are given for $s_i = -1$ (open triangles) and $s_i = 1$ (crosses). (*b*)Enlargement of panel (*a*), with all orbits up to $p = 27$ included.

Use S_{GK} to denote a symbol sequence obtained from figure 2, and S_{BW} to denote the symbol sequence of the same trajectory obtained from figure 3. Then, the translation from S_{GK} to S_{BW} is achieved by the replacements

$$0011 \to 0000 \tag{2.3}$$

$$10111111 \rightarrow 10001111 \tag{2.4}$$

$$10111110100 \rightarrow 10001110100 \tag{2.5}$$

$$1011111010111111 \rightarrow 1000111010111111 \tag{2.6}$$

$$10111110101111010 \rightarrow 10001110101111010 \tag{2.7}$$

$$\ldots \tag{2.8}$$

where the ellipsis stand for rules beyond our accuracy.

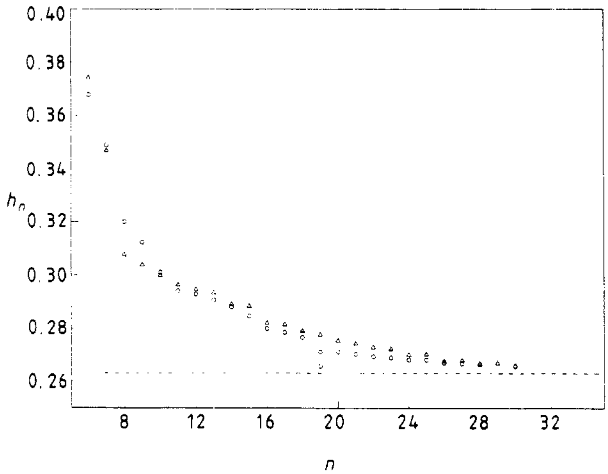

Figure 4. Metric entropy estimates h_n (equation (1.2)), obtained by using the partitions of figure 2 (open triangles) and of figure 3 (open circles). The broken line indicates the metric entropy h, estimated via the Lyapunov exponent.

Due to the special grammar of the symbol strings, this recoding is indeed invertible, i.e. it represents a non-trivial entropy preserving map between symbol sequences. Let us illustrate this just with the first replacement, 0011 $\rightarrow$ 0000. In S_{GK}, the sequence 0000 is forbidden. Thus, any such sequence appearing in S_{BW} must have been produced by this replacement. For a sequence 100001 in S_{BW}, the backtranslation is obviously unique. For 1000001, we have *a priori* the two possibilities 1000111 and 1001101. But the latter is forbidden in S_{GK}, whence the backtranslation of 1000001 is also unique. More than five 0 are forbidden in S_{BW}. For the other sequences, analogous arguments hold.

We conclude thus that the method of [11] fails for $a = 1.0$ and $b = 0.54$, but the failure is not dramatic. Even for this large value of b, (1.6) has hardly any other limit sets than fixed points, these still represent most of the periodic orbits of the Henon map, and (1.6) yields most orbits only once, attaching to them thus (nearly) unique symbol sequences. For smaller values of b ($b \leq 0.3$), it is either correct or it gives at least results which are better than those obtainable from any other method at present. We shall thus use the method in the next section to study periodic orbits of the Henon map in some more detail.

3. Applications

For a strictly hyperbolic attractor, the invariant measure can be approximated arbitrarily closely by a sum of delta measures concentrated on the periodic orbits. When using orbits of period p, the weight of each point x on a periodic orbit is [16]

$$\mu(\boldsymbol{x}) = \mathrm{e}^{-ph(\boldsymbol{x})} \tag{3.1}$$

where $h(\boldsymbol{x})$, the pointwise metric entropy, is the sum over all positive Lyapunov exponents. Notice that this is a strict equality, not just a proportionality, i.e. we have

$$\lim_{p\to\infty} \sum_{\boldsymbol{x}\in\mathrm{Fix}(p)} \mathrm{e}^{-ph(\boldsymbol{x})} = 1. \tag{3.2}$$

In our case, we have only one positive Lyapunov exponent. Introducing the generating function (or 'partition function')

$$Q_p(\beta) = \sum_{\boldsymbol{x}\in\mathrm{Fix}(p)} \mathrm{e}^{-\beta p\lambda(\boldsymbol{x})} \tag{3.3}$$

we can write this as $\lim_{p\to\infty} \ln Q_p(\beta = 1) = 0$.

It is widely believed that these relations also hold for a not strictly hyperbolic attractor of the Henon type. In order to test this, we have plotted in figure 5 the values of $Q_p(\beta = 1)$ as a function of p. We find that they seem indeed to converge towards 1, but the convergence is rather slow and irregular.

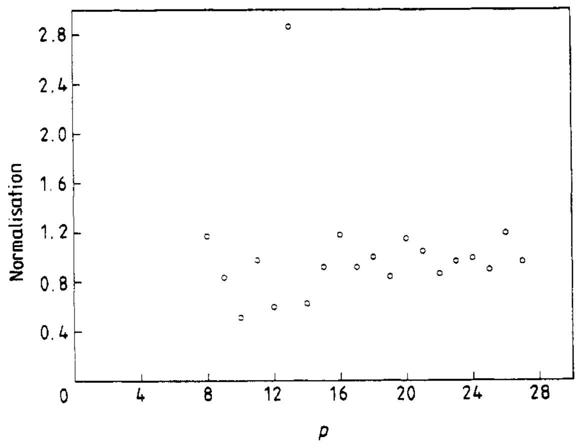

Figure 5. Normalisation of the weights of periodic orbits, i.e. $Q_p(\beta = 1)$.

Most characteristic quantities of the attractor can also be expressed in terms of $Q_p(\beta)$. For instance, the (average) Lyapunov exponent is given by

$$\lambda = \lim_{p\to\infty} \frac{1}{p} \frac{\partial Q_p(\beta)}{\partial\beta}\bigg|_{\beta=1} \tag{3.4}$$

$$= \lim_{p\to\infty} \frac{1}{p} \frac{\partial \ln Q_p(\beta)}{\partial\beta}\bigg|_{\beta=1}. \tag{3.5}$$

In figure 6, we show the right-hand sides as functions of p. Again we see that the convergence is not particularly fast.

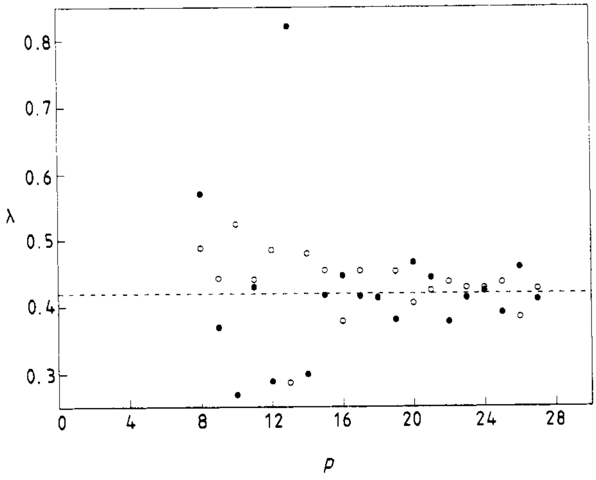

Figure 6. Lyapunov exponents obtained from periodic orbits: from equation (3.4) (open cirlces), and from equation (3.5) (full circles). The broken line represents the true value.

The strong anomalies in figures 5 and 6 at $p = 13$ are due to a pair of orbits with anomalously small Lyapunov exponents, $\lambda(\boldsymbol{x}) = 0.199$ and 0.201, respectively. These seem to be the orbits with the smallest $\lambda(\boldsymbol{x})$ among all periodic orbits. That orbits with very small Lyapunov exponents typically come in pairs is also found for other periods (e.g., at period 16 with $\lambda = 0.2600$ and 0.26195) and for one-humped maps of the interval [10]. This is easily understood: the two orbits have one point on opposite sides very close to a primary tangency (respectively to the critical point), apart from this their symbol sequences are the same.

In addition to the Lyapunov exponent, also generalised dimensions and $f(\alpha)$ functions can be obtained from $Q_p(\beta)$ [4,8,15]. We found that in most cases the convergence was not better than for the Lyapunov exponent. To illustrate this, we show in figure 7 the finite-p approximations $\Phi_p(\Lambda)$ to the scaling function denoted $\Phi(\Lambda)$ in [4]. They are obtained by Legendre transforming $\ln Q_p/p$. We see that functions for successive values of p converge only very slowly to the final asymptotic result.

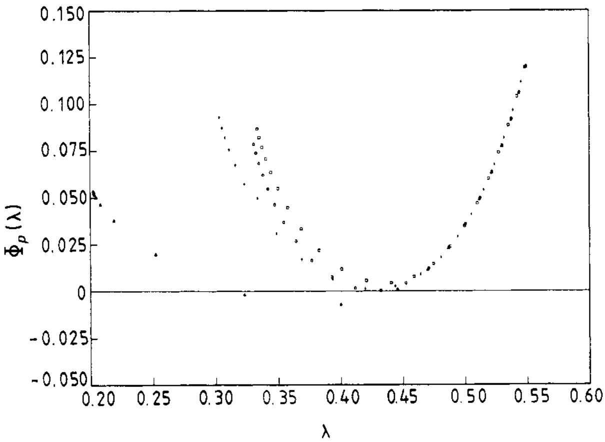

Figure 7. Scaling functions $\Phi_p(\Lambda)$ for $p = 24$ (open circles), 25 (open squares), 26 (open triangles) and 27 (open diamonds).

The convergence can be substantially improved by going over to dynamical ζ functions [17]. The ζ function is defined via the 'grand canonical partition function' Ω as

$$\Omega(z,\beta) = \sum_{p=1}^{\infty} z^p Q_p(\beta) \tag{3.6}$$

$$= \frac{z}{\zeta(z,\beta)} \frac{\partial\zeta(z,\beta)}{\partial z} \tag{3.7}$$

and $\zeta(0,\beta) = 1$. It can be rewritten [17] as an infinite product over primitive cycles:

$$\zeta(z,\beta)^{-1} = \prod_p [1 - z^p \exp(-p\beta\lambda_p)]. \tag{3.8}$$

Equations (3.2) and (3.5) are equivalent to the fact that $\zeta(z,\beta)^{-1}$ has a simple zero at $z_0(\beta)$ with $z_0(1) = 1$ and $\mathrm{d}z_0(\beta)/\mathrm{d}\beta|_{\beta=1} = \lambda$. In figures 8 and 9, we show the approximations to $z_0(1)$ (respectively to $\mathrm{d}z_0(\beta)/\mathrm{d}\beta|_{\beta=1}$) obtained from all periods up to a given p, and using the method called 'cycle expansion' in [17]. The latter consists of expanding $\zeta(z,\beta)^{-1}$ into a power series in z, and truncating the series at the highest power whose coefficient is known. If all cycles with periods $\leq p$ have been included, this power is just p.

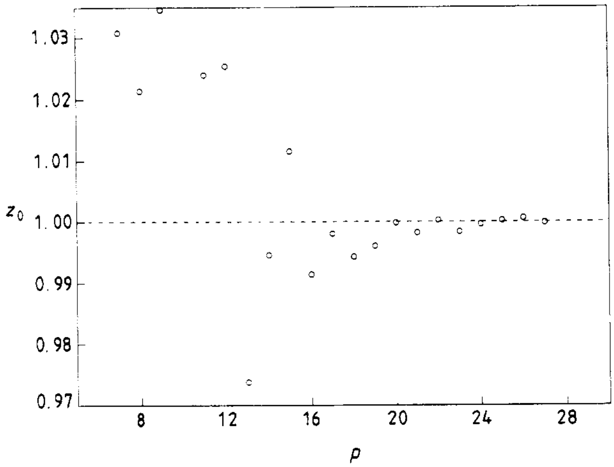

Figure 8. Zeros $z_0(\beta = 1)$ of the pth-order polynomial approximants of $\zeta^{-1}(z,\beta = 1)$.

As expected, we find much improved convergence. Still, the above casts serious doubts on the claims of [8,15] that periodic orbits provide a numerically convenient tool for studying strange attractors. Essentially, the irregularities in figures 6 and 7 show that 'curvatures' [17] are very non-negligible due to the orbits with anomalously small Lyapunov exponents, and due to irregularities in the pruning rules. Both are due to the fact that the Henon attractor is not stricly hyperbolic. A more regular behaviour can be expected for strictly hyperbolic sets with simple symbolic dynamics (like horseshoe repellers), but for physically realistic attractors strict hyperbolicity seems to be the exception rather than the rule. These findings agree with the claims of [17] that fast convergence of ζ functions can only be expected if the topological dynamics

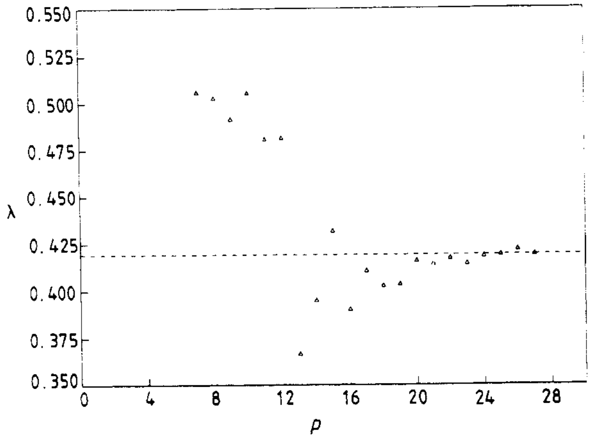

Figure 9. Lyapunov exponents obtained from $\partial z_0(\beta)/\partial\beta|_{\beta=1}$ of the pth-order polynomial expansion of $\zeta^{-1}(z,\beta)$, the broken line corresponds to the value obtained from standard methods.

is understood and a proper encoding of symbol sequences is used, thus redefining the period of orbits.

Finally, we want to comment on an observation made in [11]. These authors found that the topological entropy is constant in regions where the strange set is a repelling horseshoe. The related observation that simple grammars prevail in such regions was made in [8, 18]. In all these cases the authors seemed unaware that this is a well known phenomenon: hyperbolic horseshoes are structurally stable and thus both the grammar of their symbolic dynamics and the topological entropy are constant under small changes. This is maybe best known for quadratic maps of the iterval, where, e.g., in the entire period-3 window one has the simple grammar saying that $\ldots LL\ldots$ is forbidden, giving for the topological entropy the logarithm of the golden ratio.

4. Conclusions

For the usual parameters of the Henon map, we confirmed the surprising claims of [11], and we verified that the method proposed there gives the same symbolic dynamics as proposed earlier. The absence of any other limit sets apart from simple fixed points suggests strongly that there exist Lyapunov functions for the artificial dynamical system introduced in [11]. For $b = 0$ we have indeed found Lyapunov functions for every fixed point of (1.6) for arbitrary a, but not for $b \neq 0$.

For larger values of the paramter b in the Henon map, the method of [11] breaks down: on the one hand we find limit cycles where there should be none according to [11], on the other hand the induced symbolic dynamics is not unique and hence it does not correspond to a generating partition.

Finally, applying the method where it seems to work, we find that expressions for dimensions and Lyapunov exponents based on periodic orbits show much slower convergence than proposed originally, in agreement with more recent work.

Appendix.

In this appendix, we restrict ourselves to the case $b = 0$, i.e. the forces reduce to $f_i(\boldsymbol{x}) = x_{i+1} - a + x_i^2$.

Let $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_N)$ be a fixed point $f_i(\xi) = 0$ of (1.6), i.e. a periodic orbit of the logistic map. We then define

$$V_{\xi}(\boldsymbol{x}) = \sum_{i=1}^{N} \epsilon_i f_i^2(\boldsymbol{x}) \tag{A1}$$

with

$$\epsilon_N = 1 \tag{A2}$$

$$\epsilon_{i-1} = \epsilon_i \frac{4}{\mu^2} |\xi_i \xi_{i-1}| \tag{A3}$$

$$\mu = 2 \left| \prod_{i=1}^{N} \xi_i \right|^{1/N}. \tag{A4}$$

Notice that μ is the Lyapunov number of $\boldsymbol{\xi}$ when considered as an orbit of $x_{i+1} = a - x_i^2$, whence $\mu > 1$ for an unstable orbit.

We claim that for any such unstable orbit, $V_{\xi}(\boldsymbol{x})$ is a local Lyapunov function of (1.6), if the sequence S is chosen such that $s_i = -\mathrm{sgn}(\xi_i)$.

For a proof, we have to show that:

(i) V_{ξ} has a localminimum at the fixed point $\boldsymbol{\xi}$ of (1.6). This is obviously true since all ε are positive.

(ii) $\mathrm{d}V_{\xi}/\mathrm{d}t < 0$ in some finite neighbourhood of this fixed point.

Straightforward use of (1.6) and (1.7) with $b = 0$ gives

$$\begin{aligned} \frac{\mathrm{d}V_{\xi}}{\mathrm{d}t} &= 2\sum_{i=1}^{N} (\dot{x}_{i+1} + 2x_i \dot{x}_i) f_i \epsilon_i \\ &= 2\sum_{i=1}^{N} (s_{i+1} f_{i+1} + 2x_i f_i s_i) f_i \epsilon_i. \end{aligned} \tag{A5}$$

Using

$$2|f_{i+1} f_i| \le c_i f_i^2 + (1/c_i) f_{i+1}^2 \tag{A6}$$

for real positive c_i, we can estimate this as

$$\frac{\mathrm{d}V_{\xi}}{\mathrm{d}t} \le \sum_{i=1}^{N} \left(4x_i s_i \epsilon_i f_i^2 + \epsilon_i c_i f_i^2 + \frac{\epsilon_i}{c_i} f_{i+1}^2 \right). \tag{A7}$$

We use now that the symbol sequence S is such that $s_i = -\mathrm{sgn}(\xi_i)$. Furthermore, as is easily checked,

$$\epsilon_N = \epsilon_1 (4/mu^2) |\xi_n \xi_1| = \epsilon_0. \tag{A8}$$

Thus, we can write

$$\left.\frac{\mathrm{d}V_{\xi}}{\mathrm{d}t}\right|_{\boldsymbol{x}=\boldsymbol{\xi}} \le \sum_{i=1}^{N} \left(-4|\xi_i| + c_i + \frac{\epsilon_{i-1}}{\epsilon_i c_{i-1}} \right) \epsilon_i f_i^2 \tag{A9}$$

and all we have to show is that there exist positive c_i such that

$$4|\xi_i| > c_i + \frac{\epsilon_{i-1}}{\epsilon_i c_{i-1}}. \tag{A10}$$

Using (A3), this can be rewritten as

$$2\mu > \frac{\mu c_i}{2|\xi_i|} + \frac{2|\xi_{i-1}|}{\mu c_{i-1}}. \tag{A11}$$

It is obviously true for $c_i = 2|\xi_i|/\mu$.

The above argument does not say anything about the sizes of the neighbourhoods on which $V_\xi(\boldsymbol{x})$ is a Lyapunov function. In particular, it does not explain why these neighbourhoods seem to overlap in one common domain.

Finally, for stable orbits, $\mu < 1$, a straightforward linear stability analysis shows that equation (1.6) can converge only if $\prod_i s_i = -1$, independent of the signs of the ξ_i.

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